Dense numerical sets, Kronecker's Theorem and else...

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Introduction.

Among the problems of mathematical olympiads there are problems that by one way or another related to the approximations of irrational numbers by rational ones. Such problems directly lead to theorems of the theory of Diophantine approximations, such as the Kronecker Theorem and Dirichlet Theorem, and to the concept of subset that dense in a given set (a concept important for understanding the fundamental properties of real numbers). Thus, such problems, in addition to their competitive Olympiad assignments, become a cognitive stimulus.

These notes can be considered as a short introduction to the topic mentioned above with applications to olympiad problems.

I. **Preliminary facts related to integer and fractional parts.**

1. Let x is real number then $\{n\{mx\}\} = \{nmx\}$ for any $n, m \in \mathbb{Z}$.

Indeed, $\{n\{mx\}\} = \{n(mx - \lfloor mx \rfloor)\} = \{nmx\}$.

2. Note that for irrational τ and any integer n number $\{n\tau\}$ is irrational as well, because

otherwise $\tau = \dfrac{\{n\tau\} + [n\tau]}{n} \in \mathbb{Q}$.

3. Archimedes's Axiom:

For any real $\alpha > 0$ there is natural n such that $n\alpha > 1$.

II. A little bit theory.

For further we need some facts that represented by the following lemmas:

Lemma 1.

Each interval (α, β) with the length $\beta - \alpha > 1$ contain at least one integer number.

Proof.

Denote $n := \lfloor \alpha \rfloor + 1$. Then from $\lfloor \alpha \rfloor \leq \alpha < n$ and $\alpha + 1 < \beta$ follows $\alpha < n = \lfloor \alpha \rfloor + 1 \leq \alpha + 1 < \beta$.

Lemma 2.

Let $\tau \in (0,1)$ and $\tau \notin \mathbb{Q}$,then there are unique natural k and irrational ρ such that

$1 = k\tau + \rho$ and $0 < \rho < \tau$.

Proof.

Really, denote $k := \left\lfloor \dfrac{1}{\tau} \right\rfloor$, $\rho := \tau \left\{ \dfrac{1}{\tau} \right\}$ then from $\dfrac{1}{\tau} = \left\lfloor \dfrac{1}{\tau} \right\rfloor + \left\{ \dfrac{1}{\tau} \right\}$ we immediately obtain $1 = k\tau + \rho$ and $0 < \rho < \tau$, where $\rho \neq 0$ (because τ is irrational)

and integer $k \geq 0$ (because $0 < \tau < 1$).

Suppose that there is else $1 = k_1\tau + \rho_1$ and $0 < \rho_1 < \tau$, then $|k - k_1|\tau = |\rho - \rho_1| < \tau \iff |k - k_1| < 1 \Longrightarrow k = k_1 \Longrightarrow \rho = \rho_1$.

Lemma 3.

Let $\theta \in (0,1)$ be irrational number and k any natural number, then exist *integer* $m \neq 0$ such that

$\{m\theta\} < \dfrac{1}{k}$ and $|m| \leq k$.

Proof.

Let consider sequence $x_i := \{i\theta\}$ where $i = 1, 2, ..., k+1$. Since all these numbers are different (because otherwise if $x_i = x_j$ for some $i \neq j$ then

$\{i\theta\} = \{j\theta\} \iff i\theta - \lfloor i\theta \rfloor = j\theta - \lfloor j\theta \rfloor \iff \theta(i-j) = \lfloor i\theta \rfloor - \lfloor j\theta \rfloor \iff$

$\theta = \dfrac{\lfloor i\theta \rfloor - \lfloor j\theta \rfloor}{i-j} \in \mathbb{Q}$ and that contradict to irrationality of θ) then there are x_i and x_j such that $0 < x_i - x_j < \dfrac{1}{k}$.

Indeed, assume the contrary $|x_i - x_j| \geq \dfrac{1}{k}$ for all $i \neq j$.

Let $y_1 < y_2 < ... < y_{k+1}$ are all terms of sequence $x_1, x_2, ..., x_{k+1}$ in increasing order.

Since by assumption $y_{i+1} - y_i \geq \dfrac{1}{k}, i = 1, ..., k$ then we obtain $y_{k+1} - y_1 =$

$(y_{k+1} - y_k) + (y_k - y_{k-1}) + ... + (y_2 - y_1) \geq k \cdot \dfrac{1}{k} > 1$.

But that contradict to $0 < y_1 < y_{k+1} < 1$.

Since $x_i - x_j = \{i\theta\} - \{j\theta\} = (i\theta - \lfloor i\theta \rfloor) - (j\theta - \lfloor j\theta \rfloor) =$

$\theta(i-j) - \lfloor i\theta \rfloor + \lfloor j\theta \rfloor$ and $0 < x_i - x_j < \dfrac{1}{k}$ we obtain

$x_i - x_j = \{\theta(i-j) - \lfloor i\theta \rfloor + \lfloor j\theta \rfloor\} = \{\theta(i-j)\}$.

So, $\{m\theta\} < \dfrac{1}{k}$ for $m = i - j$ and $|m| \leq k$ because

$-k = 1 - (k+1) \leq i - j \leq (k+1) - 1 = k$.

Remark 1. Actually don't necessary to claim that $\theta \in (0,1)$, because for any irrational θ by **Lemma3** for irrational $\{\theta\} \in (0,1)$ there is *integer* $m \neq 0$ such that $\{m\{\theta\}\} < \dfrac{1}{k}$ and $\{m\{\theta\}\} = \{m\theta\}$.

Corollary1.

Let θ be irrational and k be any natural number, then there is the *natural* m such that $\{m\theta\} < \dfrac{1}{k}$.

Proof.

Suppose that m which was obtained in the **Lemma 3** is negative, then by **Lemma 2** $1 = l \cdot \{m\theta\} + \theta_1$, where $l \in \mathbb{N}$ and $0 < \theta_1 < \{m\theta\}$.

Hence $\theta_1 = \{\theta_1\} = \{1 - l \cdot \{m\theta\}\} = \{-l \cdot m\theta + l\lfloor m\theta \rfloor\} = \{-l \cdot m\theta\} = \{m_1\theta\}$, where $m_1 := -lm > 0$ and since $\theta_1 < \{m\theta\} < \dfrac{1}{k}$ we have now positive m_1 and $\{m_1\theta\} < \dfrac{1}{k}$.

(*But this* m_1 *can be greater then* k and *as the price for positivity of* m_1 *we lost* $|m| \leq k$).

Remark 2.

For any $\theta \in \mathbb{R} \diagdown \mathbb{Q}$ and any $\varepsilon > 0$ there is natural m such that $\{m\theta\} < \varepsilon$.

Indeed, since for any $\varepsilon > 0$ by Archimedes's **Axiom** there is $k \in \mathbb{N}$ that $k\varepsilon > 1 \iff \dfrac{1}{k} < \varepsilon$ then $\{m\theta\} < \dfrac{1}{k} < \varepsilon$.

Corollary 2.(Dirihlet's Theorem)

Let θ irrational and k arbitrary natural number, then exist integer m and l such that $|m\theta - l| < \dfrac{1}{k}$ and $0 < m \leq k$.

Proof.

By **Lemma 3** we have $0 < m\theta - \lfloor m\theta \rfloor < \dfrac{1}{k} \implies |m\theta - \lfloor m\theta \rfloor| < \dfrac{1}{k} \iff$ $||m|\,\theta - \lfloor m\theta \rfloor \cdot sign\,(m)| < \dfrac{1}{k}$.

Let $l := \lfloor m\theta \rfloor \cdot sign\,(m)$, $m := |m|$. Then we obtain $|m\theta - l| < \dfrac{1}{k}$ where $0 < m \leq k$.

Corollary3.

For any irrational θ and any natural k there is rational $r = \dfrac{l}{m}$ such that $|\theta - r| < \dfrac{1}{mk}$ and $0 < m \leq k$..

Corollary4.

Let θ irrational number and $\varepsilon > 0-$ arbitrary real number, then inequalities below have infinitely

many solutions:

$$\textbf{a)}\ \{x \cdot \theta\} < \varepsilon,\quad x \in \mathbb{Z}$$
$$\textbf{b)}\ \{x \cdot \theta\} < \varepsilon,\quad x \in \mathbb{N}$$
$$\textbf{c)}\ |x \cdot \theta - y| < \varepsilon,\ x \in \mathbb{N},\ y \in \mathbb{Z}.$$

Proof.

a) Inequality $\{x \cdot \theta\} < \dfrac{1}{k}$ where $k \in \mathbb{N}$ and $\dfrac{1}{k} \leq \varepsilon$ has at least one solution in $\mathbb{Z}$, which is also solution of inequality $\{x \cdot \theta\} < \varepsilon$.

Suppose that fore some $\varepsilon > 0$ inequality has finite set S of solutions , then $0 < \delta := \min\limits_{x \in S} \{x \cdot \theta\}$ and there is no solutions for inequality $\{x \cdot \theta\} < \delta$ in $\mathbb{Z}$. But that is contradiction, because for natural k , such that $\dfrac{1}{k} < \delta$, by **Corollary 1** inequality $\{x \cdot \theta\} < \dfrac{1}{k}$ has solution in $\mathbb{Z}$.

By the same way can be proved **(b)** and **(c).**

Definition.

We say that proper subset A of numerical set X dense in X if for any real $\varepsilon > 0$

and any $x \in X$ there is $a \in A$ such that $|x - a| < \varepsilon$.**(Approximation Form)**

If $X = (p, q)$ and $A \subsetneq (p, q)$ then easy to see that A is dense in (p, q) if for any subinterval $(\alpha, \beta) \subset (p, q)$ there is $a \in A$ such that $\alpha < a < \beta$.**(Interval form);**

If $A \subsetneq \mathbb{R}$ and A is dense in $\mathbb{R}$ we say that A is everywhere dense.

Immediately from definition follows:

If $A \subsetneq \mathbb{R}$ everywhere dense (dense in $\mathbb{R}$) and $\tau \in \mathbb{R} \setminus \{0\}$ then $\tau + A$ and τA dense in $\mathbb{R}$.

Proof.

Let $(\alpha, \beta) \in \mathbb{R}$.Then for interval $(\alpha - \tau, \beta - \tau)$ there is $a \in (\alpha - \tau, \beta - \tau) \iff a + \tau \in (\alpha, \beta)$

and in the case $\tau > 0$ for interval $\left(\dfrac{\alpha}{\tau}, \dfrac{\beta}{\tau}\right)$ there is $a \in \left(\dfrac{\alpha}{\tau}, \dfrac{\beta}{\tau}\right) \iff \tau a \in (\alpha, \beta)$.

If $\tau < 0$ then for interval $\left(\dfrac{\beta}{\tau}, \dfrac{\alpha}{\tau}\right)$ there is $a \in \left(\dfrac{\beta}{\tau}, \dfrac{\alpha}{\tau}\right) \iff \tau a \in (\alpha, \beta)$.

Examples.

1. Set of rational numbers $\mathbb{Q}$ is everywhere dense, that is for any two real $\alpha < \beta$ there is $r \in \mathbb{Q}$

such that $\alpha < r < \beta$.($\mathbb{Q}$ everywhere dense in $\mathbb{R}$)

Indeed, by Archimedes's Axiom there is natural n such that

$$n(\beta - \alpha) > 1 \iff n\beta - n\alpha > 1$$

and by **Lemma 1** there is $m \in \mathbb{N}$ such that $n\alpha < m < n\beta \iff \alpha < \dfrac{m}{n} < \beta$.

2. Let $A := \{\{\sqrt{n}\} \mid n \in \mathbb{N}\}$.Then A dense in $(0,1)$.

Indeed, let $(\alpha, \beta) \subset (0,1)$.We will prove that there is $m, n \in \mathbb{N}$ such that $\alpha < \sqrt{n} - m < \beta \iff \alpha + m < \sqrt{n} < \beta + m \iff (\alpha + m)^2 < n < (\beta + m)^2$.

Accordingly to **Lemma 1** we claim $(\beta + m)^2 - (\alpha + m)^2 > 1 \iff$

$(\beta - \alpha)(2m + \alpha + \beta) > 1 \iff 2m + \alpha + \beta > \dfrac{1}{\beta - \alpha} \iff$

$$m > \frac{1 - (\beta^2 - \alpha^2)}{2(\beta - \alpha)}.$$

Thus, for any natural $m > \dfrac{1 - (\beta^2 - \alpha^2)}{2(\beta - \alpha)}$ there is $n \in \mathbb{N}$ such that

$(\alpha + m)^2 < n < (\beta + m)^2 \iff \alpha + m < \sqrt{n} < \beta + m \implies$

$$m < \sqrt{n} < 1 + m \iff m = [\sqrt{n}].$$

Then $\alpha + m < \sqrt{n} < \beta + m \iff \alpha < \sqrt{n} - [\sqrt{n}] < \beta \iff \alpha < \{\sqrt{n}\} < \beta$.

Theorem (Leopold Kronecker)

a) For any irrational θ set $\{\{n\theta\} \mid n \in \mathbb{N}\}$ dense in $(0,1)$.

b) For any irrational θ set $\{n\theta + m \mid n \in \mathbb{N}, m \in \mathbb{Z}\}$ everywhere dense (dense in $\mathbb{R}$). (that is for any $a \in \mathbb{R}$ and any $\varepsilon > 0$ there are $n \in \mathbb{N}, m \in \mathbb{Z}$ that $|a - (n\theta + m)| < \varepsilon$).

Proof(Traditional)

a) Firstly, in supposition $\theta \in (0,1)$, we will prove that for any $\alpha, \beta \in [0,1]$ and $\alpha < \beta$ there is natural n such that $\alpha < \{n\theta\} < \beta$.

By **Corollary1** to **Lemma3 (Remark 2)** there is $m \in \mathbb{N}$, such that $\{m\theta\} < \beta - \alpha$.

Denote $\delta := \{m\theta\}$ and consider the sequence $0, \delta, 2\delta, ..., n\delta, ...$.Since $\beta - \alpha > \delta$ then $\dfrac{\beta}{\delta} - \dfrac{\alpha}{\delta} > 1$ and, therefore, by **lemma1** there is natural n such that $\dfrac{\alpha}{\delta} < n < \dfrac{\beta}{\delta} \iff \alpha < n\delta < \beta$.Since $n\delta \in (0,1)$ then $n\delta = \{n\delta\} = \{n\{m\theta\}\} = \{nm\theta\}$ and for $n := nm$ we get $\alpha < \{n\theta\} < \beta$.

Let now θ be any irrational number. Then $\theta_1 := \{\theta\} = \theta - [\theta]$ is irrational as well and, therefore,

there is natural n such hat $\alpha < \{n\theta_1\} < \beta \iff \alpha < \{n\theta - n[\theta]\} < \beta \iff \alpha < \{n\theta\} < \beta$.

b) First we will prove that for any interval (α, β) there are $n \in \mathbb{N}, m \in \mathbb{Z}$ that $\alpha < n\theta + m < \beta$.

WLOG we can assume that $\beta - \alpha \leq 1$.Then $(\{\alpha\}, \beta - [\alpha]) \in [0,1]$ and by **a)** there is $n \in \mathbb{N}$

such that $\{\alpha\} < \{n\theta\} < \beta - [\alpha] \iff \alpha < \{n\theta\} + [\alpha] < \beta \iff \alpha < n\theta - [n\theta] + [\alpha] < \beta \iff$

$\alpha < n\theta + m < \beta$,where $m := [\alpha] - [n\theta] \in \mathbb{Z}$.

Let a be any real number. Then for any $0 < \varepsilon$ there is $n \in \mathbb{N}$, $m \in \mathbb{Z}$ that

$a - \varepsilon < n\theta + m < a + \varepsilon \iff |a - (n\theta + m)| < \varepsilon$.

Now we will consider another, new constructive proof of Kronecker's Theorem.

The two following lemmas represent part **a)** of the theorem and also gives algorithm of finding n for any interval (α, β) or for any $\varepsilon > 0$ dependently from form of definition of density (Interval Form or Approximation Form).

Lemma 4.

For any irrational $\tau \in (0,1)$ there is natural $k \geq 2$ such that $\{k\tau\} < \dfrac{\tau}{2}$

Proof

For given τ we have representation $1 = k_0\tau + \tau_1$, where $k_0 \in \mathbb{N}$ and $0 < \tau_1 < \tau$. If $0 < \tau_1 < \dfrac{\tau}{2}$, then again (because τ_1 irrational and $\tau_1 \in (0,1)$) we have $1 = k_1\tau_1 + \tau_2$ where $k_1 \geq 2$ because $\tau_1 < \dfrac{1}{2}$ and $0 < \tau_2 < \tau_1 < \dfrac{\tau}{2}$.

Therefore $\tau_2 = \{\tau_2\} = \{1 - k_1\tau_1\} = \{-k_1(1 - k_0\tau)\} = \{k\tau\} < \dfrac{\tau}{2}$,where $k := k_0k_1 \geq 2$. If $\dfrac{\tau}{2} < \tau_1$,then from $\tau - \tau_1 = \{\tau - \tau_1\} = \{\tau - 1 + k_0\tau\} = \{(k_0 + 1)\tau\}$ follows $\{k\tau\} < \dfrac{\tau}{2}$,where $k = k_0 + 1 \geq 2$.

Lemma 5.

Let $\theta \in (0,1)$ be irrational, then there is the sequence of natural numbers $n_1 < n_2 < ... < n_k < ...$ such that $\{n_k\theta\} < \dfrac{\theta}{2^k}$.

Proof.

By **Lemma 4**. exist natural $k \geq 2$ such that $\{k\theta\} < \dfrac{\theta}{2}$.Let $n_1 := k$.

Suppose that we already have $n_1 < n_2 < ... < n_i$ such that

$$\theta_j := \{n_j\theta\} < \frac{\theta}{2^j},\ j = 1, 2, ...i.$$

Applying **Lemma 4** to irrational θ_i we obtain $\theta_{i+1} := \{k_i\theta_i\} < \dfrac{\theta_i}{2}$ for some natural $k_i \geq 2$.But $\theta_i < \dfrac{\theta}{2^i}$ and $\{k_i\theta_i\} = \{k_i\{n_i\theta\}\} = \{n_{i+1}\theta\} < \dfrac{\theta}{2^{i+1}}$ where $n_{i+1} := k_i n_i > n_i$.

Corollary 1

For any irrational $\theta \in (0,1)$ and for any positive ε exist infinitely many natural n such that $\{n\theta\} < \varepsilon$.More precisely, exist increasing sequence of natural numbers $n_1 < n_2 < ... < n_k < ...$ such that $\varepsilon > \{n_k\theta\}$ and $\{n_{k+1}\theta\} < \dfrac{\{n_k\theta\}}{2}$, $k \in \mathbb{N}$.

Proof

For $\{n\theta\}$ there is $m \geq 2$ such that $\{mn\theta\} = \{m\{n\theta\}\} < \dfrac{\{n\theta\}}{2} < \varepsilon$. Then for $n := n_k$ we obtain $n_{k+1} := mn_k > n_k$ for which $\{n_{k+1}\theta\} < \dfrac{\{n_k\theta\}}{2}$.

Corollary 2.

Let $\theta \in (0,1)$ and irrational, then set $\left\{\{n\theta\} \vdots\ n \in \mathbb{N}\right\}$ everywhere dense in the $(0,1)$.Moreover, for each interval $(\alpha,\beta) \in (0,1)$ inequality $\alpha < \{x\theta\} < \beta$ has infinitely many natural solutions.

Proof

By **Corollary 1 to Lemma 3 (Remark2)** there is natural m such that $\{m\theta\} < \beta - \alpha$.Then interval $\left(\dfrac{\alpha}{\{m\theta\}}, \dfrac{\beta}{\{m\theta\}}\right)$ has length greater then 1 and by **Lemma 1** contain at least one natural n, namely:

$\dfrac{\alpha}{\{m\theta\}} < n < \dfrac{\beta}{\{m\theta\}} \iff \alpha < n\{m\theta\} < \beta \implies \alpha < \{nm\theta\} < \beta$, because from $n\{m\theta\} \in (0,1)$ follows $n\{m\theta\} = \{n\{m\theta\}\} = \{nm\theta\}$.(For example we can take $n := \left\lfloor \dfrac{\alpha}{\{m\theta\}} \right\rfloor + 1$). So, we have one natural solution $x := nm$ of inequality $\alpha < \{x\theta\} < \beta$.

But by **Corollary 2.** always exist natural $m' > m$ such that $\{m'\theta\} < \dfrac{\{m\theta\}}{2}$.

Then $n' := \left\lfloor \dfrac{\alpha}{\{m'\theta\}} \right\rfloor + 1 > n$.(precisely $n' \geq 2n - 1$, because

$$n' - 1 = \left\lfloor \frac{\alpha}{\{m'\theta\}} \right\rfloor \geq \left\lfloor \frac{2\alpha}{\{m\theta\}} \right\rfloor \geq 2\left\lfloor \frac{\alpha}{\{m\theta\}} \right\rfloor = 2(n-1)\ \ )$$

and we get another solution natural solution $x' = m'n' > x$ of inequality $\alpha < \{x\theta\} < \beta$ and this process can be continued infinitely.

By this way starting from m and $n = \left\lfloor \dfrac{\alpha}{\{m\theta\}} \right\rfloor + 1$ we obtain infinite increasing sequence of natural solutions of inequality $\alpha < \{x\theta\} < \beta$.

For applications often convenient the following interval form of **Kronecker's Theorem**

c) Corollary 3

If $\theta \in (0,1)$ is irrational, then for any interval $(\alpha,\beta) \subset \mathbb{R}$ there are $n,m \in \mathbb{N}$ such that $\alpha < n\theta - m < \beta$ (that is $\left\{n\theta - m \vdots\, n,m \in \mathbb{N}\right\}$ everywhere dense (dense in $\mathbb{R}$).

Proof

Let $(\alpha,\beta) \subset \mathbb{R}$. Without loss generality we can suppose that $\lfloor\alpha\rfloor = \lfloor\beta\rfloor$. Since $(\{\alpha\},\{\beta\}) \subset (0,1)$ then inequality $\{\alpha\} < \{n\theta\} < \{\beta\}$ have natural solution so big as we need. In particular it has solution $n \geq \dfrac{\lfloor\alpha\rfloor + 1}{\theta}$.Since $n \geq \dfrac{\lfloor\alpha\rfloor + 1}{\theta} \iff n\theta \geq \lfloor\alpha\rfloor + 1 \implies [n\theta] \geq \lfloor\alpha\rfloor + 1 \iff [n\theta] - \lfloor\alpha\rfloor \geq 1$ then denoting $m := [n\theta] - \lfloor\alpha\rfloor$ we obtain

$$\alpha < \{n\theta\} < \{\beta\} \iff \alpha - \lfloor\alpha\rfloor < n\theta - \lfloor n\theta\rfloor < \beta - \lfloor\alpha\rfloor \iff$$
$$\alpha < n\theta - (\lfloor n\theta\rfloor - \lfloor\alpha\rfloor) < \beta \iff \alpha < n\theta - m < \beta.$$

III. Applications to problems.

Problem 1

Prove that for any natural M with k digits there is natural n such that first k digits of 2^n is precisely M.

Solution

Assertion of problem can be written as: exist $m \in \mathbb{N} \cup \{0\}$ such that

$M = \left\lfloor \dfrac{2^n}{10^m} \right\rfloor \iff M \leq \dfrac{2^n}{10^m} < M + 1 \iff \log M \leq n\log 2 - m < \log(M+1)$.

So we have to prove that this inequality has natural solution n and m.

But that immediately follows from **Corollary 7** for $\theta := \log 2$.

Possible directly solution which based on **Corollary 4(b)** to the **Lemma 3.**

Because M is k digits number then $\lfloor\log M\rfloor = k$.Denote $\alpha := \{\log M\} = \log M - k$ and $\beta := \min\{1, \log(M+1) - k\}$, then $(\alpha,\beta) \subset (0,1)$ .By Corollary 4(b) to Lemma 3 inequality $\{x\log 2\} < \beta - \alpha$ has infinitely many natural solutions.Let take one of them greater than

$$\frac{k}{\log 2}, \text{ i.e. let } n > \frac{k}{\log 2} \text{ and } \{n\log 2\} < \beta - \alpha$$

then interval $\left(\dfrac{\alpha}{\{n\log 2\}}, \dfrac{\beta}{\{n\log 2\}}\right)$ has length greater than 1 and consequently contain natural l .So we get inequality

$\alpha < l\{n\log 2\} < \beta \iff \log M < ln\log 2 - (l\lfloor n\log 2\rfloor - k) < \beta + k \leq \log(M+1)$.

Denote $m := l\lfloor n\log 2\rfloor - k$ and $n := ln$ then we obtain inequality

$\log M \leq n \log 2 - m < \log (M+1)$ where $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$.

Problem 2.

Prove that exist irrational θ such that set $\{2^n \theta \mid n \in \mathbb{N}\}$ is everywhere dense in [0,1).

Solution.

Let all natural numbers are represented in the binary system. So we have $\mathbb{N} = \{1, 10, 11, 100, 101, 110, 111, 1000, ...\}$

and let θ is real number in which after dot consecutively was written all natural numbers in binary notation. By the other words

$\theta = .11011100101110111100001001....$ This number is irrational because its binary representation contains zero segments of any length.

This number has interesting property:

For each number $b = 0.\beta_1\beta_2...\beta_k$ we can find natural number which indicate the position in θ from where digits of number b starting as segment digits of

θ.Therefore, we can define function $l(b)$ which shows the least number of starting positions of number b.For example if $\theta = 0.\theta_1\theta_2....\theta_m...$, then

$$\left\{2^{l(b)}\theta\right\} = 0.\beta_1\beta_2...\beta_k\theta_{l(b)+k+1}...$$

Let $a = 0.\alpha_1\alpha_2...\alpha_i... \in (0,1)$ and let p arbitrary natural , then for $b := 2^{-p}\lfloor 2^p\alpha \rfloor = 0.\alpha_1\alpha_2...\alpha_p$ numbers α and $\left\{2^{l(b)}\theta\right\}$ have the same first p digits – $\alpha_1, \alpha_2, ...\alpha_p$. Therefore $\left|\alpha - \left\{2^{l(b)}\theta\right\}\right| = 2^{-p}\left|0.\alpha_{p+1}... - 0.\theta_{l(b)+p+1}...\right| \leq 2^{-p}$.

Problem 3.

Set $S := \{\sin n \mid n \in \mathbb{N}\}$ is dense in $[-1, 1]$.

Solution.

Note that $\dfrac{n}{2\pi} = \left[\dfrac{n}{2\pi}\right] + \left\{\dfrac{n}{2\pi}\right\}$ implies

$$\sin n = \sin\left(2\pi\left[\frac{n}{2\pi}\right] + 2\pi\left\{\frac{n}{2\pi}\right\}\right) = \sin\left(2\pi\left\{\frac{n}{2\pi}\right\}\right).$$

Since $\left\{n \cdot \dfrac{1}{2\pi}\right\}$ is dense in $[0,1)$ then $2\pi\left\{\dfrac{n}{2\pi}\right\}$ is dense in $[0, 2\pi)$.

To complete the proof it suffices to apply for $\sin x$ the following

Proposition.

Let f is continuous on $[a,b]$ and $f([a,b]) = [m,M]$ and $A \subsetneq [a,b]$ dense in $[a,b]$

then $f(A)$ dense in $[m,M]$.

Proof.

Let $q \in [m,M]$ and $f(p) = q$ for some $p \in [a,b]$. For any $\varepsilon > 0$ there is $\delta > 0$ such that $|x - p| < \delta \implies |f(x) - q| < \varepsilon$.

Since A dense in $[a,b]$ then there is $a \in A$ such that $|a - p| < \delta$.Then $|f(a) - q| < \varepsilon$ and that mean $f(A)$ dense in $[m,M]$.

Problem 4. Let $A := \{\{\log n\} \mid n \in \mathbb{N}\}$.Then A dense in $(0,1)$.

Hint. $B := \{\{\log n\} \mid n = 2^m, m \in \mathbb{N}\} = \{\{m \log 2\} \mid m \in \mathbb{N}\} \subset A$ and B dense in $(0,1)$.

Problem 5.
Between two rational numbers always lie at least one irrational.
Hint. Consider the interval $\left(r_1/\sqrt{2}, r_2/\sqrt{2}\right)$.
And more problems to solve.
Problem 6.
a) Let $A := \{\sqrt{n} - \sqrt{m} \mid n, m \in \mathbb{N}\}$ .Then A everywhere dense.
b) Let $A := \{\sqrt[3]{n} - \sqrt{m} \mid n, m \in \mathbb{N}\}$ .Then A everywhere dense.
Problem 7.
Prove that set $\left\{r^3 \mid r \in \mathbb{Q}\ \right\}$ is everywhere dense.
Problem 8.
Prove that set $\left\{\ln\left(r^2+1\right) \mid r \in \mathbb{Q}\ \right\}$ is dense in $[0, \infty)$.
Problem 9.
Prove that for any natural $q \geq 2$ there is such real number α that any interval $(a, b) \subset (0, 1)$
contain at least one term of the sequence $a_n = \{\alpha q^n\}, n \in \mathbb{N}$.

